MARKOV CHAINS ON ORTHOGONAL BLOCK STRUCTURES

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ABSTRACT. In this paper we define a particular Markov chain on some combinatorial structures called orthogonal block structures. These structures include, as a particular case, the poset block structures, which can be naturally regarded as the set on which the generalized wreath product of permutation groups acts as the group of automorphisms. In this case, we study the associated Gelfand pairs together with the spherical functions.

1. INTRODUCTION

This paper takes origin from an analysis of the article [3], where the generalized wreath product of permutation groups is introduced. This group can be regarded as the group of automorphisms of a certain poset called poset block structure. This construction contains, as a particular case, the action of the classical permutation wreath product on the rooted tree. In this case, considering the full automorphism group of the tree and the subgroup fixing a leaf, one gets a Gelfand pair. The associated decomposition into irreducible submodules can be alternatively obtained by the spectral analysis of a Markov chain on the set of the leaves of the tree (see [7] and [6]). The idea is to extend the Markov chain to any poset block structure. In particular, we will prove that the generalized wreath product and a subgroup fixing a vertex of the poset block associated are still a Gelfand pair. The decomposition into irreducible submodules is given in [3], we find the corresponding spherical functions and the relative eigenvalues. Actually, the group structure is not essential to define the Markov chain, we only need the poset block structure. This suggests considering a more general combinatorial structure, known as orthogonal block structure, defined in [8]. This is a collection $F$ of uniform partitions of a finite set satisfying some orthogonality conditions. In this case, the Markov chain can be defined only using the set $F$ of partitions. This is the motivation for starting our analysis of orthogonal block structures and
then for focusing our attention on the poset block structures and their groups of automorphisms. Our construction recalls the Markov chain introduced in [4] in the different context of lattices associated with the semigroup of ordered partitions of a finite set, belonging to a particular class of semigroups called left-regular bands. Our Markov chain is defined on a finite set Ω and it is induced by the simple random walk on a poset associated with a special family of unordered partitions of Ω constituting an orthogonal block structure. We also give an original interpretation from the Gelfand pairs theory point of view, in relation with the action of generalized wreath products of groups on poset block structures.

2. Orthogonal block structures

2.1. Preliminaries. The following definitions can be found in [2]. Given a partition $F$ of a finite set $\Omega$, let $R_F$ be the relation matrix of $F$, i.e.

$$R_F(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are in the same part of } F \\ 0 & \text{otherwise.} \end{cases}$$

If $R_F(\alpha, \beta) = 1$, we usually write $\alpha \sim_F \beta$.

**Definition 2.1.** A partition $F$ of $\Omega$ is **uniform** if all its parts have the same size. This number is denoted $k_F$.

The trivial partitions of $\Omega$ are the universal partition $U$, which has a single part and the equality partition $E$, all of whose parts are singletons. We denote by $J_\Omega$ and $I_\Omega$ their relation matrices, respectively.

The partitions of $\Omega$ constitute a poset with respect to the relation $\preceq$, where $F \preceq G$ if every part of $F$ is contained in a part of $G$. We use $F \prec G$ if $F \preceq G$ and $F \preceq H \preceq G$ implies $H = F$ or $H = G$. Given any two partitions $F$ and $G$, their infimum is denoted $F \wedge G$ and it is the partition whose parts are intersections of $F$–parts with $G$–parts; their supremum is denoted $F \vee G$ and it is the partition whose parts are minimal subject to being unions of $F$–parts and $G$–parts.

**Definition 2.2.** A set $\mathcal{F}$ of uniform partitions of $\Omega$ is an **orthogonal block structure** if:

1. $\mathcal{F}$ contains $U$ and $E$;
2. for all $F$ and $G \in \mathcal{F}$, $\mathcal{F}$ contains $F \wedge G$ and $F \vee G$;
3. for all $F$ and $G \in \mathcal{F}$, the matrices $R_F$ and $R_G$ commute with each other.

2.2. Probability. Let $\mathcal{F}$ be an orthogonal block structure on the finite set $\Omega$. We want to associate with $\mathcal{F}$ a Markov chain on $\Omega$. In order to perform this, we define a new poset $(P, \preceq)$ starting from the partitions in $\mathcal{F}$.

Let $C = \{E = F_0, F_1, \ldots, F_n = U\}$ be a maximal chain of partitions in $\mathcal{F}$ such that $F_i \preceq F_{i+1}$ for all $i = 0, \ldots, n - 1$. Let us define a rooted
tree of depth \( n \) as follows: the \( n \)–th level is constituted by \(|\Omega|\) vertices; the \((n - 1)\)–st by \([|\Omega|]_{kF_1}\) vertices. Each of these vertices is a father of \( k_{F_1} \) sons that are in the same \( F_1 \)–class. Inductively, at the \( i \)–th level there are \([|\Omega|]_{k_{F_{n-i}}}/k_{F_{n-i-1}}\) vertices which are fathers of \( k_{F_{n-i}} \) vertices of the \((i + 1)\)–st level representing \( F_{n-i-1} \)–classes contained in the same \( F_{n-i} \)–class.

We can perform the same construction for every maximal chain \( C \) in \( \mathcal{F} \). The next step is to glue the different rooted trees associated with each maximal chain by identifying the vertices associated with the same partition. The resulting structure is the poset \((P, \leq)\).

**Example 2.3.**

Consider the set \( \Omega = \{000, 001, 010, 011, 100, 101, 110, 111\} \) and the set of partitions of \( \Omega \) given by \( \mathcal{F} = \{E, F_1, F_2, F_3, U\} \) where, as usually, \( E \) denotes the equality partition and \( U \) the universal partition of \( \Omega \). The nontrivial partitions are defined as:

- \( F_1 = \{000, 001, 010, 011\} \coprod \{100, 101, 110, 111\} \);
- \( F_2 = \{000, 001\} \coprod \{010, 011\} \coprod \{100, 101\} \coprod \{110, 111\} \);
- \( F_3 = \{000, 010\} \coprod \{001, 011\} \coprod \{100, 110\} \coprod \{101, 111\} \).

So the orthogonal block structure \( \mathcal{F} \) can be represented as the following poset:

![Diagram of the poset](image)

**Fig.1.** The orthogonal block structure \( \mathcal{F} = \{E, F_1, F_2, F_3, U\} \). The maximal chains in \( \mathcal{F} \) have length 3 and they are:

- \( C_1 = \{E, F_2, F_1, U\} \);
- \( C_2 = \{E, F_3, F_1, U\} \).

The associated rooted trees \( T_1 \) and \( T_2 \) have depth 3 and they are, respectively,
So the poset \((P, \leq)\) associated with \(\mathcal{F}\) is

Observe that, if \(F_1 \subsetneq F_2\), then the number of \(F_1\)-classes contained in a \(F_2\)-class is \(k_{F_2}/k_{F_1}\).

The Markov chain we want to describe, is performed on the last level of the poset \((P, \leq)\) associated with the set \(\mathcal{F}\). We can think of an insect which, at the beginning of our process, lies on a fixed element \(\omega_0\) of \(\Omega\) (this corresponds to the identity relation \(E\), i.e. each element is in relation only with itself). The insect randomly moves reaching an adjacent vertex in \((P, \leq)\) (this corresponds, in the orthogonal block structure \(\mathcal{F}\), to move from \(E\) to another relation \(F\) such that \(E \subsetneq F\), i.e. \(\omega_0\) is identified with all the elements in the same \(F\)-class) and so on. At each step in \((P, \leq)\) (that does not correspond necessarily to a step in the Markov chain on \(\Omega\)) the insect could randomly move from the \(i\)-th level of \((P, \leq)\) either to the \((i-1)\)-st level or to the \((i+1)\)-st level. Going up means to pass in \(\mathcal{F}\) from a partition \(F\) to a partition \(L\) such that \(F \subsetneq L\) (these are \(|\{L \in \mathcal{F} : F \subsetneq L\}|\) possibilities in \((P, \leq))\), going down means to pass in \(\mathcal{F}\) to a partition \(J\) such that \(J \subsetneq F\) (these are \(\sum_{J \in \mathcal{F}, J \subsetneq F} \frac{k_F}{k_J}\) possibilities in \((P, \leq))\). The next step of the random
walk is whenever the insect reaches once again the last level in \((P, \leq)\).

In order to formalize this idea let us introduce the following definitions.

Let \(F \triangleleft G\) and let \(\alpha_{F,G}\) be the probability of moving from the partition \(F\) to the partition \(G\). So the following relation is satisfied:

\[
\alpha_{F,G} = \frac{1}{\sum_{J \in \mathcal{F} : J \triangleleft F} (k_F/k_J) \alpha_J,F \alpha_{F,G}} + \sum_{J \in \mathcal{F} : J \triangleleft F} \sum_{L \in \mathcal{F} : J \triangleleft F} (k_F/k_J) \alpha_{J,L} \alpha_{F,G} \sum_{J \in \mathcal{F} : J \triangleleft F} (k_F/k_J) + |\{L \in \mathcal{F} : F \triangleleft L\}|.
\]

In fact, the insect can directly pass from \(F\) to \(G\) with probability \(\alpha_{F,G}\) or go down to any \(J\) such that \(J \triangleleft F\) and then come back to \(F\) with probability \(\alpha_{J,F}\) and one starts the recursive argument. From direct computations one gets

\[
\alpha_{E,F} = \frac{1}{|\{L \in \mathcal{F} : E \triangleleft L\}|},
\]

where \(E\) denotes the equality partition. Moreover, if \(\alpha_{E,F} = 1\) we have, for all \(G\) such that \(F \triangleleft G\)

\[
\alpha_{F,G} = \frac{1}{\sum_{J \in \mathcal{F} : J \triangleleft F} (k_F/k_J) + |\{L \in \mathcal{F} : F \triangleleft L\}|};
\]

if \(\alpha_{E,F} \neq 1\), the coefficient \(\alpha_{F,G}\) is defined as in (1).

**Definition 2.4.** For every \(\omega \in \Omega\), we define

\[
p(\omega_0, \omega) = \sum_{E \neq F \in \mathcal{F}} \sum_{\omega_0 \sim F \omega} \frac{\alpha_{E,F_1} \cdots \alpha_{F',F} \left(1 - \sum_{F \triangleleft L} \alpha_{F,L}\right)}{k_F}.
\]

The fact that \(p\) is effectively a transition probability on \(\Omega\) will follow from Theorem 2.7. For each partition \(F \neq E, U\), we define the following numbers:

\[
p_F = \sum_{C \subseteq \mathcal{F} \text{ chain}} \alpha_{E,F_1} \cdots \alpha_{F',F} \left(1 - \sum_{F \triangleleft L} \alpha_{F,L}\right).
\]

Observe that \(p_F\) expresses the probability of reaching the partition \(F\) but no partition \(L\) such that \(F \triangleleft L\) in \(\mathcal{F}\). Moreover, we put

\[
p_U = \sum_{C \subseteq \mathcal{F} \text{ chain}} \alpha_{E,F_1} \cdots \alpha_{F',U}.
\]

The coefficients \(P_F\) constitute a probability distribution on \(\mathcal{F} \setminus \{E\}\), as the following lemma shows.
Lemma 2.5. The coefficients $p_F$ defined in (4) and (5) satisfy the following identity:

$$\sum_{E \not\in F} p_F = 1.$$  

Proof. Using the definitions, we have

$$\sum_{E \not\in F} p_F = \sum_{E \not\in F, F \not\in U} \sum_{C \subseteq F \text{ chain}} \alpha_{E,F_1} \cdots \alpha_{F',F} \left(1 - \sum_{F \not\in L} \alpha_{F,L}\right) + p_U$$

$$= \sum_{E \not\in F} \alpha_{E,F} = 1.$$  

In fact, for every $F \in \mathcal{F}$ such that $E \not\in F \neq U$, given a chain $C = \{E, F_1, \ldots, F', F\}$ we get the terms $\alpha_{E,F_1} \cdots \alpha_{F',F} \left(1 - \sum_{F \not\in L} \alpha_{F,L}\right)$. Since $C = \{E, F_1, \ldots, F', F, L\}$ is still a term of the sum one can check that only the summands $\sum_{E \not\in F} \alpha_{E,F}$ are not cancelled. The thesis follows from (2).

For every $F \in \mathcal{F}, F \neq E$, we define $M_F$ as the Markov operator whose transition matrix is

$$M_F = \frac{1}{k_F} R_F.$$  

(6)

Definition 2.6. Given the operators $M_F$ as in (6) and the coefficients $p_F$ as in (4) and (5), set

$$M = \sum_{E \not\in F} p_F M_F.$$  

(7)

By abuse of notation, we denote by $M$ the stochastic matrix associated with the Markov operator $M$.

Theorem 2.7. $M$ coincides with the transition matrix of $p$.

Proof. By direct computation we get:

$$M(\omega_0, \omega) = \sum_{E \not\in F} p_F M_F(\omega_0, \omega) = \sum_{E \not\in F} p_F \frac{1}{k_F}$$

$$= \sum_{E \not\in F} \sum_{\omega_0 \sim F \omega} \alpha_{E,F_1} \cdots \alpha_{F',F} \left(1 - \sum_{F \not\in L} \alpha_{F,L}\right) \frac{1}{k_F}$$

$$= p(\omega_0, \omega).$$

□
2.3. **Spectral analysis of** $M$. We give here the spectral analysis of the operator $M$ acting on the space $L(\Omega)$ of the complex functions defined on the set $\Omega$ endowed with the scalar product $\langle f_1, f_2 \rangle = \sum_{\omega \in \Omega} f_1(\omega)\overline{f_2(\omega)}$. First of all (see, for example, [1]) we introduce, for every $F \in \mathcal{F}$, the following subspaces of $L(\Omega)$:

$$V_F = \{ f \in L(\Omega) : f(\alpha) = f(\beta) \text{ if } \alpha \sim_F \beta \}.$$  

It is easy to show that the operator $M_F$ defined in (6) is the projector onto $V_F$. In fact if $f \in L(\Omega)$, then $M_F f(\omega_0)$ is the average of the values that $f$ takes on the elements $\omega$ such that $\omega \sim_F \omega_0$ and so $M_F f = f$ if $f \in V_F$ and $M_F f = 0$ if $f \in V_F^\perp$.

Set

$$W_G = V_G \cap \left( \sum_{G \prec F} V_F \right)^\perp.$$  

In [1] it is proven that $L(\Omega) = \bigoplus_{G \in \mathcal{F}} W_G$. We can deduce the following proposition.

**Proposition 2.8.** The $W_G$’s are eigenspaces for the operator $M$ whose associated eigenvalue is

$$\lambda_G = \sum_{E \neq F \in \mathcal{F}} p_F.$$  

**Proof.** By definition, $W_G \subseteq V_G$. This implies that, if $f \in W_G$,

$$M_F f = \begin{cases} f & \text{if } F \preceq G \\ 0 & \text{otherwise} \end{cases}$$

So, for $w \in W_G$, we get

$$M \cdot w = \sum_{E \neq F \in \mathcal{F}} p_F M_F \cdot w$$

$$= \left( \sum_{E \neq F \in \mathcal{F}} p_F \right) \cdot w.$$  

Hence the eigenvalue $\lambda_G$ associated with the eigenspace $W_G$ is

$$\lambda_G = \sum_{E \neq F \in \mathcal{F}} p_F,$$  

and the assertion follows. \qed

**Example 2.9.**

We want to study the transition probability $p$ in the case of the orthogonal block structure of the Example 2.3. One can easily verify that:
\[ \alpha_{E,F_2} = \alpha_{E,F_3} = \alpha_{F_2,F_1} = \alpha_{F_3,F_1} = \frac{1}{2}; \]
\[ \alpha_{F_1,U} = \frac{1}{3}. \]

Let us compute the transition probability \( p \) on the last level of \((P, \leq)\):

\[
\begin{pmatrix}
17 & 11 & 11 & 5 & 1 & 1 & 1 & 1 \\
11 & 17 & 5 & 1 & 1 & 1 & 1 & 1 \\
11 & 5 & 17 & 11 & 1 & 1 & 1 & 1 \\
11 & 5 & 17 & 11 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 17 & 11 & 11 & 5 \\
1 & 1 & 1 & 1 & 11 & 17 & 5 & 11 \\
1 & 1 & 1 & 1 & 11 & 5 & 17 & 11 \\
1 & 1 & 1 & 1 & 5 & 11 & 11 & 17
\end{pmatrix}
\]

The corresponding transition matrix is given by

\[
P = \frac{1}{48}
\begin{pmatrix}
17 & 11 & 11 & 5 & 1 & 1 & 1 & 1 \\
11 & 17 & 5 & 1 & 1 & 1 & 1 & 1 \\
11 & 5 & 17 & 11 & 1 & 1 & 1 & 1 \\
11 & 5 & 17 & 11 & 1 & 1 & 1 & 1 \\
5 & 11 & 11 & 17 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 17 & 11 & 11 & 5 \\
1 & 1 & 1 & 1 & 11 & 17 & 5 & 11 \\
1 & 1 & 1 & 1 & 11 & 5 & 17 & 11 \\
1 & 1 & 1 & 1 & 5 & 11 & 11 & 17
\end{pmatrix}
\]

The coefficients \( p_F \), with \( E \neq F \), are the following:

\[
\begin{align*}
p_U &= 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}; \\
p_{F_1} &= 2 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3}; \\
p_{F_2} &= \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}; \\
p_{F_3} &= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.
\end{align*}
\]
The Markov operator $M$ is given by (see (7) and (6)):  

$$M = \frac{1}{4}M_{F_2} + \frac{1}{4}M_{F_3} + \frac{1}{3}M_{F_1} + \frac{1}{6}M_U$$

and its eigenvalues, according with formula (8), are the following:

- $\lambda_U = 1$
- $\lambda_{F_1} = \frac{5}{6}$
- $\lambda_{F_2} = \frac{1}{4}$
- $\lambda_{F_3} = \frac{1}{4}$
- $\lambda_E = 0$

3. The case of poset block structures

A particular class of orthogonal block structures is given by the so called poset block structures.

3.1. Preliminaries. Let $(I, \leq)$ be a finite poset, with $|I| = n$. First of all, we need some definitions (see, for example, [3]).

**Definition 3.1.** A subset $J \subseteq I$ is said

- **ancestral** if, whenever $i > j$ and $j \in J$, then $i \in J$;
- **hereditary** if, whenever $i < j$ and $j \in J$, then $i \in J$;
- **a chain** if, whenever $i, j \in J$, then either $i \leq j$ or $j \leq i$;
- **an antichain** if, whenever $i, j \in J$ and $i \neq j$, then neither $i \leq j$ nor $j \leq i$.

In particular, for every $i \in I$, the following subsets of $I$ are ancestral:

$$A(i) = \{j \in I : j > i\} \quad \text{and} \quad A[i] = \{j \in I : j \geq i\},$$

and the following subsets of $I$ are hereditary:

$$H(i) = \{j \in I : j < i\} \quad \text{and} \quad H[i] = \{j \in I : j \leq i\}.$$

Given a subset $J \subseteq I$, we set

- $A(J) = \bigcup_{i \in J} A(i)$;
- $A[J] = \bigcup_{i \in J} A[i]$;
- $H(J) = \bigcup_{i \in J} H(i)$;
- $H[J] = \bigcup_{i \in J} H[i]$.

**Lemma 3.2.** There exists a one-to-one correspondence between antichains and ancestral subsets of $I$.

**Proof.** First of all we prove that, given an antichain $S$, the set $A_S = I \setminus H[S]$ is ancestral. Assuming $i \in A_S$ and $j > i$, then it must be $j \in A_S$. In fact, if $j \in H[S]$, then we should have $i \in H(S)$, since $i < j$; this is a contradiction.

Now let us show that this correspondence is injective. Suppose that, given two antichains $S_1$ and $S_2$, with $S_1 \neq S_2$, one gets $A_{S_1} = A_{S_2}$. This implies that $H[S_1] = H[S_2]$. By hypothesis we can suppose without loss of generality that there exists $s_1 \in S_1 \setminus (S_1 \cap S_2)$. Hence $s_1 \in H(S_2)$ and
there exists \( s_2 \in S_2 \) such that \( s_1 < s_2 \). So \( s_2 \in H[S_1] \). In particular, if \( s_2 \in S_1 \) we find a contradiction because \( S_1 \) is an antichain; if \( s_2 \in H(S_1) \) there exists \( s'_1 \in S_1 \) such that \( s'_1 > s_2 > s_1 \), and then we have a contradiction again.

That is why the map \( S \rightarrow I \setminus H[S] \), for each antichain \( S \), is injective.

Given an ancestral set \( J \), we define the set of the maximal elements in \( I \setminus J \) as \( S_J = \{ i \in I \setminus J : A(i) \cap (I \setminus J) = \emptyset \} \). It is easy to prove that \( S_J \) is an antichain. In fact if \( i, j \in S_J \) then, if \( i < j \) or \( i > j \), we can surely say that one of \( i \) or \( j \) is not maximal.

Now we show that \( J = I \setminus H[S_J] \). This is equivalent to show that \( I \setminus J = H[S_J] \). First we have that \( I \setminus J \subseteq H[S_J] \) because if \( i \) is maximal in \( I \setminus J \) than it belongs to \( S_J \), otherwise there exists \( j \in S_J \) such that \( i < j \), and so \( i \in H[S_J] \). On the other hand, let \( i \) be in \( H[S_J] \). If \( i \) is in \( S_J \), then it is in \( I \setminus J \) by definition. If \( i \) is in \( H(S_J) \) there exists \( j \) in \( S_J \) such that \( i < j \). Furthermore if \( i \) is an element of \( J \) then \( j \) has the same property since \( J \) is ancestral and this is absurd and so \( H[S_J] \subseteq I \setminus J \). This shows that \( J = I \setminus H[S_J] \).

From what said above we have the required bijective correspondence

\[
S \leftrightarrow I \setminus H[S]
\]

between antichains and ancestral sets. \( \square \)

**Remark 3.3.**

Note that, for \( S = \emptyset \), one gets \( A_S = I \).

In what follows we will use the notation in [3].

For each \( i \in I \), let \( \Delta_i = \{ \delta_0, \ldots, \delta_{i-1} \} \) be a finite set, with \( m \geq 2 \).

For \( J \subseteq I \), put \( \Delta_J = \prod_{i \in J} \Delta_i \). In particular, we put \( \Delta = \Delta_I \).

If \( K \subseteq J \subseteq I \), let \( \pi_K^J \) denote the natural projection from \( \Delta_J \) onto \( \Delta_K \). In particular, we set \( \pi_J = \pi_J^I \) and \( \delta_J = \delta \pi_J \). Moreover, we will use \( \Delta^J_i \) for \( \Delta^{A(i)}_J \) and \( \pi_i^J \) for \( \pi_i^{A(i)} \).

Let \( A \) be the set of ancestral subsets of \( I \). If \( J \in A \), then the equivalence relation \( \sim_J \) on \( \Delta \) associated with \( J \) is defined as

\[
\delta \sim_J \epsilon \iff \delta_J = \epsilon_J,
\]

for each \( \delta, \epsilon \in \Delta \).

**Definition 3.4.** A poset block structure is a pair \( (\Delta, \sim_A) \), where

1. \( \Delta = \prod_{i \in I} \Delta_i \), with \( (I, \leq) \) a finite poset and \( |\Delta_i| \geq 2 \), for each \( i \in I \);

2. \( \sim_A \) denotes the set of equivalence relations on \( \Delta \) defined by the ancestral subsets of \( I \).

In particular, the set \( \sim_A \) defines an orthogonal block structure on \( \Delta \).
Remark 3.5.

Note that all the maximal chains in $\mathcal{A}$ have the same length $n$. In fact, the empty set is always ancestral. A singleton $\{i\}$ constituted by a maximal element in $I$ is still an ancestral set. Inductively, if $J \in \mathcal{A}$ is an ancestral set, then $J \cup \{i\}$ is an ancestral set if $i$ is a maximal element in $I \setminus J$. So every maximal chain in the poset of ancestral subsets has length $n$. In particular, the empty set $\emptyset$ corresponds to the universal partition $U$ and $I$ to the equality partition $E$ in $\sim_A$.

Remark 3.6.

Pay attention that the operator $M_J := M_{\sim J}$ can be obtained as follows:

$$M_J = \left( \bigotimes_{i \in I \setminus H[J]} I_i \right) \otimes \left( \bigotimes_{i \in H[J]} U_i \right),$$

where $I_i$ denotes the identity operator on $\Delta_i$ and $U_i$ is the uniform operator on $\Delta_i$, whose adjacency matrix is $\frac{1}{m} I_i$.

3.2. The generalized wreath product. We present here the definition of generalized wreath product given in [3]. We will follow the same notation of the action to the right presented there. For each $i \in I$, let $G_i$ be a permutation group on $\Delta_i$ and let $F_i$ be the set of all functions from $\Delta_i$ into $G_i$. For $J \subseteq I$, we put $F_J = \prod_{i \in J} F_i$ and set $F = F_I$. An element of $F$ will be denoted $f = (f_i)$, with $f_i \in F_i$.

Definition 3.7. For each $f \in F$, the action of $f$ on $\Delta$ is defined as follows: if $\delta = (\delta_i) \in \Delta$, then

$$\delta f = \varepsilon,$$

where $\varepsilon = (\varepsilon_i) \in \Delta$ and $\varepsilon_i = \delta_i(\delta \pi^i f_i)$.

It is easy to verify that this is a faithful action of $F$ on $\Delta$. If $(I, \leq)$ is a finite poset, then $(F, \Delta)$ is a permutation group, which is called the generalized wreath product of the permutation groups $(G_i, \Delta_i)_{i \in I}$ and denoted $\prod_{(I, \leq)} (G_i, \Delta_i)$.

Definition 3.8. An automorphism of a poset block structure $(\Delta, \sim_A)$ is a permutation $\sigma$ of $\Delta$ such that, for every equivalence $\sim_J$ in $\sim_A$, $\delta \sim_J \varepsilon \iff (\delta \sigma) \sim_J (\varepsilon \sigma)$, for all $\delta, \varepsilon \in \Delta$.

The following fundamental theorems are proven in [3]. We denote by $\text{Sym}(\Delta_i)$ the symmetric group acting on the set $\Delta_i$. Later on in this paper, you can find it denoted by $\text{Sym}(m)$ if $|\Delta_i| = m$ as well $\text{Sym}(\Delta_i)$.

Theorem 3.9. The generalized wreath product of the permutation groups $(G_i, \Delta_i)_{i \in I}$ is transitive on $\Delta$ if and only if $(G_i, \Delta_i)$ is transitive for each $i \in I$. 
Theorem 3.10. Let \((\Delta, \sim_A)\) be a poset block structure with associated poset \((I, \leq)\). Let \(F\) be the generalized wreath product \(\prod_{(I, \leq)} \text{Sym}(\Delta_i)\). Then \(F\) is the group of automorphisms of \((\Delta, \sim_A)\).

Remark 3.11.

If \((I, \leq)\) is a finite poset, with \(\leq\) the identity relation, then the generalized wreath product becomes the permutation direct product:

\[
\begin{array}{cccccc}
& 1 & 2 & 3 & \cdots & n \\
1 & & & & & \\
2 & & & & & \\
3 & & & & & \\
\cdots & & & & & \\
n & & & & & \\
n-1 & & & & & \\
\end{array}
\]

In this case, we have \(A(i) = \emptyset\) for each \(i \in I\) and so an element \(f\) of \(F\) is given by \(f = (f_i)_{i \in I}\), where \(f_i\) is a function from a singleton \(\{\ast\}\) into \(G_i\) and so its action on \(\delta_i\) does not depend on any other components of \(\delta\).

Remark 3.12.

If \((I, \leq)\) is a finite chain, then the generalized wreath product becomes the permutation wreath product

\[
(G_n, \Delta_n) \wr (G_{n-1}, \Delta_{n-1}) \cdots \wr (G_1, \Delta_1).
\]

In this case, we have \(A(i) = \{1, 2, \ldots, i-1\}\) for each \(i \in I\) and so an element \(f\) of \(F\) is given by \(f = (f_i)_{i \in I}\), with

\[
f_i : \Delta_1 \times \cdots \times \Delta_{i-1} \rightarrow G_i
\]

and so its action on \(\delta_i\) depends on \(\delta_1, \ldots, \delta_{i-1}\). The Markov chain \(p\) in this case corresponds to the classical Markov chain on the ultrametric space given by the \(n\)–th level of the rooted \(q\)–ary tree studied in [7] (see also chapter 9 in [6] and [5]).

3.3. Gelfand pairs. In what follows we suppose \(G_i = \text{Sym}(m)\), where \(m = |\Delta_i|\). Fixed an element \(\delta_0 = (\delta_1^0, \ldots, \delta_n^0)\) in \(\Delta\), the stabilizer \(\text{Stab}_F(\delta_0)\) is the subgroup of \(F\) acting trivially on \(\delta_0\). If we represent \(f \in F\) as the \(n\)–tuple \((f_1, \ldots, f_n)\), with \(f_i : \Delta^i \rightarrow \text{Sym}(m)\) and we set \(\Delta_0^i = \prod_{j \in A(i)} \delta_j^0\), we have the following lemma.
Lemma 3.13. The stabilizer of $\delta_0 = (\delta_0^1, \ldots, \delta_0^n) \in \Delta$ in $F$ is the subgroup

$$K := \text{Stab}_F(\delta_0) = \{ g = (f_1, \ldots, f_n) \in F : f_i|_{\Delta_0^i} \in \text{Stab}_{\text{Sym}(m)}(\delta_0^i) \}$$

whenever $\Delta^i = \Delta_0^i$ or $A(i) = \emptyset$.

Proof. One can easily verify that $K$ is a subgroup of $F$. If $i \in I$ is such that $A(i) = \emptyset$ then, by definition of generalized wreath product, it must be $f_i(*) \in \text{Stab}_{\text{Sym}(m)}(\delta_0^i)$. For the remaining indices $i$ we have

$$\delta_0^i f = \delta_0^i \iff \delta_0^i(\delta_0^{A(i)})f_i = \delta_0^i \iff (\delta_0^{A(i)})f_i \in \text{Stab}_{\text{Sym}(m)}(\delta_0^i) \iff f_i|_{\Delta_0^i} \in \text{Stab}_{\text{Sym}(m)}(\delta_0^i).$$

□

Now we study the $K$–orbits on $\Delta$. We recall that the action of $\text{Sym}(m-1) \cong \text{Stab}_{\text{Sym}(m)}(\delta_0^0)$ on $\Delta_0$ has two orbits, namely $\{\delta_0^0\}$ and $\Delta_0 \setminus \{\delta_0^0\}$, so that $\Delta_0 = \{\delta_0^0\} \coprod (\Delta_0 \setminus \{\delta_0^0\})$. Set $\Delta_0^0 = \{\delta_0^0\}$ and $\Delta_0^1 = \Delta_0 \setminus \{\delta_0^0\}$.

Lemma 3.14. The $K$–orbits on $\Delta$ have the following structure:

$$\left( \prod_{i \in I \setminus H[S]} \Delta_0^i \right) \times \left( \prod_{i \in S} \Delta_0^i \right) \times \left( \prod_{i \in H(S)} \Delta_i \right),$$

where $S$ is any antichain in $I$.

Proof. First of all suppose that $\delta, \epsilon \in \left( \prod_{i \in I \setminus H[S]} \Delta_0^i \right) \times \left( \prod_{i \in S} \Delta_0^i \right) \times \left( \prod_{i \in H(S)} \Delta_i \right)$, for some antichain $S$. Then $\delta|_{I \setminus H[S]} = \epsilon|_{I \setminus H[S]} = \delta_0^{i|_{I \setminus H[S]}}$. If $s \in S$ we have $A(s) \subseteq I \setminus H[S]$ and this implies $(A(s))f_s \in \text{Stab}_{\text{Sym}(m)}(\delta_0^0)$. So $\epsilon_s = \delta_s(\delta_0^{A(s)}f_s)$. If $i \in H(S)$ then $A(i) \neq \emptyset$ and $\Delta^i \neq \Delta_0^0$. This implies $(A(i))f_i \in \text{Sym}(m)$ and so $\epsilon_i = \delta_i(\delta_0^{A(i)}f_i)$. This shows that $K$ acts transitively on each subset $\left( \prod_{i \in I \setminus H[S]} \Delta_0^i \right) \times \left( \prod_{i \in S} \Delta_0^i \right) \times \left( \prod_{i \in H(S)} \Delta_i \right)$ of $\Delta$.

On the other hand, let $S \neq S'$ be two distinct antichains and $\delta \in \left( \prod_{i \in I \setminus H[S]} \Delta_0^i \right) \times \left( \prod_{i \in S} \Delta_0^i \right) \times \left( \prod_{i \in H(S)} \Delta_i \right)$ and $\epsilon \in \left( \prod_{i \in I \setminus H[S']} \Delta_0^i \right) \times \left( \prod_{i \in S} \Delta_0^i \right) \times \left( \prod_{i \in H(S')} \Delta_i \right)$. Suppose $s \in S \setminus (S \cap S')$ and so $I \setminus H[S] \neq I \setminus H[S']$. If $s \in I \setminus H[S]$ then $\delta_s = \epsilon_s = \delta_s$. But $(A(S))f_s \in \text{Stab}_{\text{Sym}(m)}(\delta_0^0)$ and so $\delta_s(A(S)f_s) \neq \epsilon_s$. If $s \in H(S')$ there exists $s' \in S' \setminus (S \cap S')$ such that $s < s'$. This implies that $s' \in I \setminus H[S]$ and we can proceed as above.

The proof follows from the fact that the orbits are effectively a partition of $\Delta$. □
Finally, we prove that the group $F = \prod_{i \in I} G_i$ acting on $\Delta$ and the stabilizer $K$ of the element $\delta_0 = (\delta_0^1, \ldots, \delta_0^n)$ yield a Gelfand pair (see [5] or [6] for the definition). To show this, we use the Gelfand condition.

**Proposition 3.15.** Given $\delta, \epsilon \in \Delta$, there exists an element $g \in F$ such that $\delta g = \epsilon$ and $\epsilon g = \delta$.

**Proof.** Let $i$ be in $I$ such that $A(i) = \emptyset$. Then, by the $m$–transitivity of the symmetric group, there exists $g_i \in Sym(\Delta_i)$ such that $\delta_i g_i = \epsilon_i$ and $\epsilon_i g_i = \delta_i$. For every index $i$ such that $A(i) \neq \emptyset$ define $f_i : \Delta_i \rightarrow Sym(\Delta_i)$ as $\delta_i f_i = \epsilon_i$ and $\epsilon_i f_i = \delta_i$. So the element $g \in F$ that we get is the requested automorphism. \qed

According to what previously said we get the following corollary.

**Corollary 3.16.** $(G, K)$ is a symmetric Gelfand pair.

### 3.4. Spherical functions.

Set $L(\Delta) = \{ f : \Delta \rightarrow \mathbb{C} \}$. It is known from [3] that the decomposition of $L(\Delta)$ into $G$–irreducible submodules is given by

$$L(\Delta) = \bigoplus_{S \subseteq I \text{ antichain}} W_S$$

with

$$W_S = \left( \bigotimes_{i \in A(S)} L(\Delta_i) \right) \otimes \left( \bigotimes_{i \in S} V_i^1 \right) \otimes \left( \bigotimes_{i \in I \setminus A[S]} V_i^0 \right).$$

(11)

Here, for each $i = 1, \ldots, n$, we denote $L(\Delta_i)$ the space of the complex valued functions on $\Delta_i$, whose decomposition into $G_i$–irreducible submodules is

$$L(\Delta_i) = V_i^0 \bigoplus V_i^1,$$

with $V_i^0$ the subspace of the constant functions on $\Delta_i$ and $V_i^1 = \{ f : \Delta_i \rightarrow \mathbb{C} : \sum_{x \in \Delta_i} f(x) = 0 \}$.

**Proposition 3.17.** The spherical function associated with $W_S$ is

$$\phi_S = \bigotimes_{i \in A(S)} \varphi_i \otimes \psi_i \otimes \varrho_i,$$

(12)

where $\varphi_i$ is the function defined on $\Delta_i$ as

$$\varphi_i(x) = \begin{cases} 1 & x = \delta_i^0 \\ 0 & \text{otherwise} \end{cases},$$

$\psi_i$ is the function defined on $\Delta_i$ as

$$\psi_i(x) = \begin{cases} 1 & x = \delta_i^0 \\ -1 & \text{otherwise} \end{cases},$$

and $\varrho_i$ is the function on $\Delta_i$ such that $\varrho_i(x) = 1$ for every $x \in \Delta_i$. 

Proof. It is clear that $\phi_S \in W_S$ and $(\delta_0)\phi_S = 1$, so we have to show that each $\phi_S$ is $K$-invariant.

Set $B_1 = \{i \in A(S) : A(i) = \emptyset\}$. If there exists $i \in B_1$ such that $\delta_i \neq \delta_0$, then $(\delta)\phi_S = (\delta)\phi^k_S = 0$ for every $k \in K$, since $\delta_i \varphi_i = (\delta_i k^{-1}) \varphi_i = 0$. Hence $\phi$ and $\phi^k$ coincide on $\delta \in \Delta$ satisfying this property. So we can suppose that $\delta_i = \delta_0$ for each $i \in B_1$.

Let $B_2$ be the set of maximal elements in $A(S) \setminus B_1$. If there exists $j \in B_2$ such that $\delta_j \neq \delta_0$, then $(\delta)\phi_S = (\delta)\phi^k_S = 0$ for every $k \in K$, since $\delta_j \varphi_j = (\delta_j k^{-1}) \varphi_j = 0$. Hence $\phi$ and $\phi^k$ coincide on the elements $\delta \in \Delta$ satisfying this property. For these reasons we can suppose that $\delta_j = \delta_0$ for each $j \in B_2$. Inductively it remains to show that $(\delta)\phi_S = (\delta)\phi^k_S$ only for the elements $\delta$ such that $\delta_{A(S)} = \delta_{A(S)}^k$, i.e. $(\delta_i)\psi_i = (\delta_i)\psi^k_i$ for every $i \in S$. This easily follows from the definition of $K$ and of the function $\psi_i$.

By considering the action of $M$ on the spherical function $\phi_S$ and by using (9), we get the following eigenvalue $\lambda_S$ for $\phi_S$:

$$(13) \quad \lambda_S = \sum_{\emptyset \neq S, J : S \subseteq I \cap [S]} p_{\sim J}.$$  

3.5. The end of the story. One can note that the eigenspaces and the corresponding eigenvalues have been indexed by the antichains of the poset $I$ in (11) and in (13), but in Proposition 2.8 they are indexed by the relations of the orthogonal poset block $F$. The correspondence is the following.

Given a partition $G \in F$, it can be regarded as an ancestral relation $\sim_J$, for some ancestral subset $J \subseteq I$. Set

$$S = \{i \in J : H(i) \cap J = \emptyset\}.$$  

It is clear that $S$ is an antichain of $I$. From the definition it follows that

$$A(S) = J \setminus S \text{ and } I \setminus A[S] = I \setminus J.$$  

The corresponding eigenspace $W_S$ is:

$$W_S = \left( \bigotimes_{i \in J \setminus S} L(\Delta_i) \right) \otimes \left( \bigotimes_{i \in S} V^1_i \right) \otimes \left( \bigotimes_{i \in I \setminus J} V^0_i \right).$$  

It is easy to check that the functions in $W_S$ are constant on the equivalence classes of the relation $\sim_J$. Moreover, these functions are orthogonal to the functions which are constant on the equivalence classes of the relation $\sim_{J'}$, with $\sim_{J'} \triangleright \sim_J$ (where $J'$ is obtained from $J$ deleting an element of $S$). Since the orthogonality with the functions constant on $\sim_{J'}$ implies the orthogonality with all functions constant on $\sim_L$,
where $\sim_L \preceq \sim_J$, then we have $W_S \subseteq W_G$. On the other hand, it is easy to verify that
\[
\dim(W_S) = \dim(W_G) = m^{|J\setminus S|} \cdot (m - 1)^{|S|},
\]
and so we have $W_S = W_G$.

Analogously, if $G = \sim_J$, from (13) we get
\[
\lambda_S = \sum_{\emptyset \neq S_K : S \subseteq I \setminus H[S_K]} p_{\sim_K} = \sum_{I \setminus K : S \subseteq K} p_{\sim_K},
\]
since $S_K = \{i \in I \setminus K : A(i) = \emptyset\}$ and $H[S_K] = I \setminus K$ whose consequence is $I \setminus H[S_K] = K$. Moreover, since $S \subseteq K$ if and only if $J \subseteq K$, we get
\[
\lambda_S = \sum_{I \setminus K : J \subseteq K} p_{\sim_K} = \sum_{E \neq K : J \subseteq K \preceq \sim_J} p_{\sim_K} = \lambda_G.
\]

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